Convective Couette-type flows under condition of slip and heating at the lower boundary

Cite as: AIP Conference Proceedings **2176**, 030024 (2019); https://doi.org/10.1063/1.5135148
Published Online: 19 November 2019

V. V. Privalova, and E. Yu. Prosviryakov

Convective Couette-Type Flows Under Condition of Slip and Heating at the Lower Boundary

V. V. Privalova[1, 2, a)] and E. Yu. Prosviryakov[1, 2, b)]

[1]*Institute of Engineering Science, Ural Branch of the Russian Academy of Sciences, 34 Komsomolskaya St., Ekaterinburg, 620049, Russia*
[2]*B. N. Yeltsin Ural Federal University, 19 Mira St., Ekaterinburg, 620002, Russia*

[a)]Corresponding author: valentprival@gmail.com
[b)]evgen_pros@mail.ru

Abstract. A Couette type boundary value problem is considered for a new exact solution of the layered convection problem. The obtained solution describes the flow of a viscous incompressible fluid layer with nonzero temperature and pressure gradients along the longitudinal (horizontal) coordinates. The horizontal velocity components depend only on the vertical (transverse) coordinate of the fluid layer. The Navier slip condition and nonzero temperature gradients are specified on the lower absolutely solid boundary of the layer. The tangential stresses and constant (atmospheric) pressure are specified at the upper boundary. The possibility the occurrence of countercurrent regions and the corresponding changes in the tangential stresses and the vorticity vector are shown for the obtained particular exact solution.

PROBLEM STATEMENT

A convective Couette-type flow of a viscous incompressible fluid is described by an equations system that includes the Navier-Stokes equations in the Boussinesq approximation, the heat equation, and the incompressibility equation. We write this system in projections on the axis of the Cartesian coordinate system [1] as

$$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}=-\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right),$$

$$\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}=-\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2}\right)$$

$$\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}=-\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2}\right)+g\beta T\,, \tag{1}$$

$$\frac{\partial T}{\partial t}+V_x\frac{\partial T}{\partial x}+V_y\frac{\partial T}{\partial y}+V_z\frac{\partial T}{\partial z}=\chi\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right),$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0\,.$$

Here, V_x, V_y, and V_z are velocities parallel to the corresponding coordinate axes of a Cartesian coordinate system $Oxyz$; $P=P(x,y,z)$ is pressure deviation from hydrostatic, referred to constant average fluid density ρ; T is deviation from the average temperature; ν and χ are the kinematic viscosity and thermal diffusivity coefficients of

Mechanics, Resource and Diagnostics of Materials and Structures (MRDMS-2019)
AIP Conf. Proc. 2176, 030024-1–030024-5; https://doi.org/10.1063/1.5135148
Published by AIP Publishing. 978-0-7354-1923-0/$30.00

the fluid, respectively; g is acceleration of gravity; β is the temperature coefficient of volume expansion of the fluid.

The flow of an infinite horizontal layer of a viscous incompressible fluid is considered. The lower and upper boundaries of the layer are parallel to the Oxy plane. The fluid layer motion is assumed stationary, i.e. the functions describing velocities, temperature, and pressure are time-independent. We assume that the flow of the fluid layer is layered; therefore, there is no motion of the fluid particles from one layer to another, and the vertical velocity component is zero. Thus, the solution of system (1) is sought in the following form [2–4]:

$$
\begin{gathered}
V_x = U(z),\ V_y = V(z),\ V_z = 0, \\
P = P_0(z) + P_1(z)x + P_2(z)y, \\
T = T_0(z) + xT_1(z) + yT_2(z).
\end{gathered} \tag{2}
$$

We substitute the exact solution class (2) into system (1). The resulting system has the following form:

$$
\begin{gathered}
\nu\frac{\partial^2 V_x}{\partial z^2} = P_1,\ \nu\frac{\partial^2 V_y}{\partial z^2} = P_2, \\
\frac{\partial P_0}{\partial z} + \frac{\partial P_1}{\partial z}x + \frac{\partial P_2}{\partial z}y = g\beta\left(T_0 + T_1 x + T_2 y\right), \\
UT_1 + VT_2 = \chi\left(\frac{\partial^2 T_0}{\partial z^2} + \frac{\partial^2 T_1}{\partial z^2}x + \frac{\partial^2 T_2}{\partial z^2}y\right).
\end{gathered} \tag{3}
$$

In view of the adopted solution form (2), the incompressibility equation is satisfied identically. All the required solution functions U, V, P_0, P_1, P_2, T_0, T_1, and T_2 depend only on the vertical z-coordinate; therefore, all partial derivatives with respect to this variable in system (3) are denoted by a prime. We apply the method of indefinite coefficients and write out the coefficients at the variables x, y and the free terms. Thus, we write system (3) in the order of the further integration:

$$
\begin{gathered}
T_1'' = 0,\ P_1' = g\beta T_1,\ \nu U'' = P_1, \\
T_2'' = 0,\ P_2' = g\beta T_2,\ \nu V'' = P_2, \\
\chi T_0'' = UT_1 + VT_2,\ P_0' = g\beta T_0.
\end{gathered} \tag{4}
$$

GENERAL SOLUTION

The general solution of system (4) has the form

$$
T_1 = zC_1 + C_2,\ P_1 = \frac{g\beta}{2}\left(C_1 z^2 + 2C_2 z\right) + C_3,\ U = \frac{g\beta}{24\nu}\left(C_1 z^4 + 4C_2 z^3\right) + \frac{C_3}{2\nu}z^2 + C_4 z + C_5,
$$

$$
T_2 = zC_6 + C_7,\ P_2 = \frac{g\beta}{2}\left(C_6 z^2 + 2C_7 z\right) + C_8,\ V = \frac{g\beta}{24\nu}\left(C_6 z^4 + 4C_7 z^3\right) + \frac{z^2}{2\nu}C_8 + zC_9 + C_{10},
$$

$$
\begin{gathered}
T_0 = \frac{g\beta z^7}{1008\nu\chi}\left(C_1^2 + C_6^2\right) + \frac{g\beta z^6}{144\nu\chi}\left(C_1 C_2 + C_6 C_7\right) + \\
+\frac{z^5}{120\nu\chi}\left[g\beta\left(C_2^2 + C_7^2\right) + 3\left(C_1 C_3 + C_6 C_8\right)\right] + \frac{z^4}{24\nu\chi}\left[C_2 C_3 + C_7 C_8 + 2\nu\left(C_1 C_4 + C_6 C_9\right)\right] +
\end{gathered} \tag{5}
$$

$$+\frac{z^3}{6\chi}\left(C_2C_4+C_1C_5+C_7C_9+C_6C_{10}\right)+\frac{z^2}{2\chi}\left(C_2C_5+C_7C_{10}\right)+zC_{11}+C_{12},$$

$$P_0=\frac{g^2\beta^2z^8}{8064\nu\chi}\left(C_1^2+C_6^2\right)+\frac{g^2\beta^2z^7}{1008\nu\chi}\left(C_1C_2+C_6C_7\right)+$$

$$+\frac{gz^6\beta}{720\nu\chi}\left[g\beta\left(C_2^2+C_7^2\right)+3\left(C_1C_3+C_6C_8\right)\right]+\frac{gz^5\beta}{120\nu\chi}\left[C_2C_3+C_7C_8+2\nu\left(C_6C_9+C_1C_1\right)\right]+$$

$$+\frac{gz^4\beta}{24\chi}\left(C_2C_4+C_1C_5+C_7C_9+C_6C_{10}\right)+\frac{gz^3\beta}{6\chi}\left(C_2C_5+C_7C_{10}\right)+\frac{1}{2}gz^2\beta C_{11}+gz\beta C_{12}+C_{13}.$$

Here, the integration constants C_i $\left(i=\overline{1,13}\right)$ are determined by setting the boundary conditions.

THE EXACT SOLUTION OF THE COUETTE-TYPE BOUNDARY VALUE PROBLEM UNDER THE NAVIER SLIP CONDITION AND HEATING AT THE LOWER BOUNDARY

We write the boundary conditions for finding the exact solution to system (4). Let the lower boundary of the infinite horizontal fluid layer be given by the equation of the plane $z=0$. It is absolutely solid and motionless. Heating and the Navier slip condition are specified [5–9] at the lower boundary. Taking into account the structure of the selected generalized solution class (2), we write these conditions in the following form:

$$T_0\left(0\right)=0,\ T_1\left(0\right)=A,\ T_2\left(0\right)=B,$$
$$\alpha\left.\frac{\partial U}{\partial z}\right|_{z=0}=U\left(0\right),\ \alpha\left.\frac{\partial V}{\partial z}\right|_{z=0}=V\left(0\right). \tag{6}$$

Here, α is a dimensional slip factor (slip length).

Let the upper boundary be free and given by the equation of the plane $z=h$. The value of the pressure function is determined at this boundary. We note that the gradients of the pressure function along the horizontal (longitudinal) coordinates x and y are equal to zero at the upper boundary. It allows us to classify the considered flow as a Couette-type flow [10]. The temperature at the upper boundary is equal to the reference zero value. Besides, zero tangential stresses are specified at the upper boundary of the fluid layer. According to the structure of the solution class (2), these conditions take the form

$$T_0\left(h\right)=T_1\left(h\right)=T_2\left(h\right)=0,$$
$$P_0\left(h\right)=S,\ P_1\left(h\right)=P_2\left(h\right)=0, \tag{7}$$
$$\left.\frac{\partial U}{\partial z}\right|_{z=h}=0,\ \left.\frac{\partial V}{\partial z}\right|_{z=h}=0.$$

Thus, a particular exact solution to system (4) satisfying the boundary conditions (6) and (7) is polynomial and has the following form:

$$U=\frac{Ag\beta}{24h\nu}\left[-6h^2z^2+4hz^3-z^4+4h^3\left(z+\alpha\right)\right],$$
$$V=\frac{Bg\beta}{24h\nu}\left[-6h^2z^2+4hz^3-z^4+4h^3\left(z+\alpha\right)\right],$$

$$T_0 = \frac{\left(A^2+B^2\right)g\beta}{1008h^2\nu\chi}\left(2h^2-3hz+z^2\right)\left[-4h^4+7h^2z^2-4hz^3+z^4-2h^3\left(3z+14\alpha\right)\right]z\,, \tag{8}$$

$$T_1 = A\left(1-\frac{z}{h}\right), \quad T_2 = B\left(1-\frac{z}{h}\right),$$

$$P_0 = S+\frac{g^2\beta^2\left(A^2+B^2\right)}{8064h^2\nu\chi}\left(h-z\right)^2\left[11h^6+15h^2z^4-6hz^5+z^6-4h^3z^2\left(5z+14\alpha\right)+h^5\left(22z+56\alpha\right)+h^4z\left(z+112\alpha\right)\right],$$

$$P_1 = -\frac{Ag\beta}{2h}\left(h-z\right)^2, \qquad P_2 = -\frac{Bg\beta}{2h}\left(h-z\right)^2.$$

We can reduce the considered two-dimensional boundary value problem to a one-dimensional problem using a non-degenerate linear transformation of the rotation of the angle $\varphi = \arctan\frac{A}{B}$ [11].

The analysis of the obtained solution (8) has shown that the velocity components U and V are monotonic functions and that they do not have any stagnation points in the interval $(0;h)$ taking into account the positive slip length α (Fig. 1)

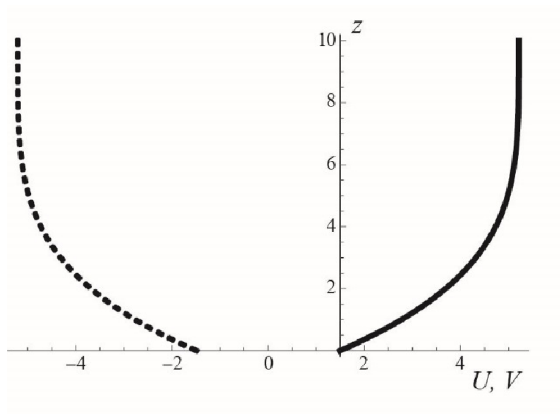

FIGURE 1. The profiles of the velocity components U (solid line) and V (dashed line) for the following parameter values: $g = 9.8$ m/s^2, $\beta = 1.82\cdot10^{-4}$ 1/K, $\nu = 10^{-6}$ m^2/s, $h = 10$ m, $\alpha = 1$ m, $A = 5\cdot10^{-4}$ K/m, $B = -5\cdot10^{-4}$ K/m

The obtained solution for the velocity in the considered fluid layer have no stagnation points, this being caused by the presence of a temperature gradient on only one of the fluid layer surfaces.

The horizontal (longitudinal) temperature gradients T_1 and T_2 are first-order polynomials whose monotonicity is determined by the conditions specified on the lower boundary. The homogeneous part of the temperature function T_0 (background temperature), which is zero at the upper and lower boundaries, reaches its minimum inside the fluid layer (Fig. 2).

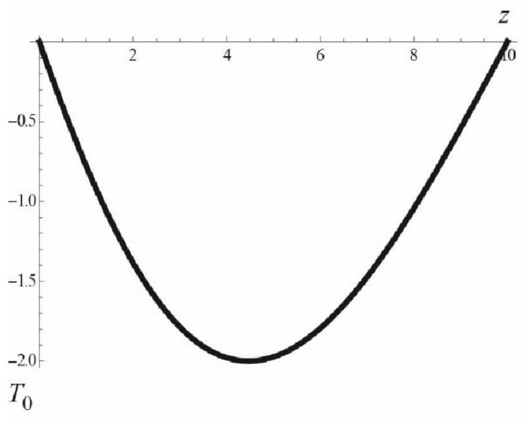

FIGURE 2. The graph of the function T_0 for the following parameter values: $g = 9.8$ m/s^2, $\beta = 1.82 \cdot 10^{-4}$ 1/K, $\nu = 10^{-6}$ m^2/s, $h = 10$ m, $\alpha = 1$ m, $A = 5 \cdot 10^{-4}$ K/m, $B = -5 \cdot 10^{-4}$ K/m

CONCLUSION

An exact solution for the two-dimensional problem of epy convective flow of a viscous incompressible fluid has been obtained. The Navier slip conditions and nonzero temperature gradients on the solid boundary have been considered for obtaining a solution to the boundary value problem. Zero tangential stresses and zero temperature and pressure gradients are specified at the free boundary. It has been demonstrated that, if temperature gradients are set only on one of the boundaries of the considered fluid layer, the two-dimensional problem can be reduced to one-dimensional one by rotation transformation.

REFERENCES

1. G. Z. Gershuni and E. M. Zhukhovitskii, *Convective Stability of Incompressible Fluids* (Keter, Jerusalem, 1976).
2. N. V. Burmasheva and E. Yu. Prosviryakov, J. Samara State Tech. Univ., Ser. Phys. Math. Sci. **21** (1), 180–196 (2017).
3. V. V. Privalova and E. Yu. Prosviryakov, J. Samara State Tech. Univ., Ser. Phys. Math. Sci. **22** (3), 1–17 (2018).
4. S. N. Aristov and E. Yu. Prosviryakov, Fluid Dynamics **51** (2), 148–154 (2016).
5. C. Neto, D. Evans, and E. Bonaccurso, Reports on Progress in Physics **39**, 2859–2897 (2005).
6. V. I. Yankov, V. I. Boyarchenko, V. P. Pervadchuk, I. O. Glot, and N. V. Shakirov, *Recycling Fiber-forming Polymers. Basics of the Polymer Rheology and the Flow of Polymers in the Channels* (R&C Dynamics Publ., Moscow, Izhevsk, 2008).
7. C. L. M. H. Navier, Mém. Acad. Sci. Inst. France **6** (2), 389–440 (1823).
8. C. L. M. H. Navier, Mém. Acad. Sci. Inst. France **7** (2), 375 (1821).
9. E. I. Borzenko, O. A. Diakova, and G. R. Shrager, Vestnik TSU. Mathematics & Mechanics **2** (28), 35–44 (2014).
10. S. N. Aristov, V. V. Privalova, and E. Yu. Prosviryakov, Rus. J. Nonlin. Dyn. **12** (2), 167–178 (2016).
11. S. N. Aristov and E. Yu. Prosviryakov, Rus. J. Nonlin. Dyn. **9** (4), 651–657 (2013).